LOOP OBSERVABLES FOR BF THEORIES IN ANY DIMENSION AND THE COHOMOLOGY OF KNOTS

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ABSTRACT. A generalization of Wilson loop observables for BF theories in any dimension is introduced in the Batalin–Vilkovisky framework. The expectation values of these observables are cohomology classes of the space of imbeddings of a circle. One of the resulting theories discussed in the paper has only trivalent interactions and, irrespective of the actual dimension, looks like a 3-dimensional Chern–Simons theory.

1. INTRODUCTION

Knot invariants can be obtained as expectation values of Wilson loops (i.e., traces of holonomies) in Chern–Simons theory [24].

The same result can be obtained in the 3-dimensional BF theory "with a cosmological term" [25, 7].

The nice feature of BF theory [21, 19], as opposed to Chern–Simons theory, is that it can be defined in any dimension and always has a quadratic term around which one can start a perturbative expansion.

On the other hand, apart from the 3-dimensional case [10, 7, 5, 6], it is rather difficult to find nontrivial observables for BF theories (see [15, 7, 12, 17] for the search of surface observables in the 4-dimensional case).

However, the formulae for the perturbative expansion of the expectation value of a Wilson loop in the 3-dimensional Chern–Simons theory (e.g., in the approach of [4]) allow for a natural generalization in any dimension [9]. The invariants defined in this way are cohomology classes of Vassiliev finite type [22] on the space of imbeddings of S^1 into $\mathbb{R}^n$. As in the 3-dimensional case, they are moreover related to certain "graph cohomologies," as originally suggested by Kontsevich [20].

This led us to look for loop observables for BF theories in any dimension whose expectation values should yield the above invariants.

In this paper we introduce these observables "on shell" (i.e., upon using the equation of motions) and describe some results about their off-shell extension (which relies on the use of the Batalin–Vilkovisky [2] formalism). For all technical details, as well as for the complete proofs of the four Theorems contained in this paper, we refer to [14].

A very interesting feature of the simplest of these observables (as well as of the cohomology classes described in [9]) is that, despite of the dimension, they are always *as if we were dealing with the 3-dimensional BF theory with a cosmological term*, which is tantamount to considering the Chern–Simons theory. This is our interpretation of Witten's ideas about a Chern–Simons theory for strings [23].

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2. BF THEORIES

The n-dimensional BF theory is a topological quantum field theory defined in terms of a connection 1-form A over a principal G-bundle $P \to M$, with $\dim M = n$, and a tensorial $(n-2)$-form B of the ad type.

Following the ideas of [12], we begin with a geometrical description of gauge invariant functionals of A and B taking values in the free loop space LM.

The basic functional is of course the Wilson loop $\text{Tr}_\rho \text{Hol}$. Here we follow the following *nonstandard*

Convention. $\text{Hol}(\gamma; A)$ denotes the group element associated to the A-parallel transport along the loop $\gamma : [0, 1] \to M$ from $\gamma(1)$ to $\gamma(0)$. (The usual holonomy is the inverse of our Hol.)

We then associate to each smooth loop γ in M and to any connection A the A-horizontal lift of γ: $[0, 1] \ni t \mapsto \gamma_A(t)$.

If we saturate the $(n-2)$-form B with the tangent vector $\dot\gamma_A(t)$, we obtain an $(n-3)$-form $B(\dot\gamma_A(t))$ defined along the horizontal path γ_A and hence an $(n-3)$-form on LM, depending on the connection A.

Then, for any representation ρ of G, the following object is well-defined and gauge invariant in any dimension:

$$(2.1) \quad h_{k,\rho}(\gamma; A, B) = \int\limits_{0 < t_1 < \ldots < t_k < 1} \text{Tr}_\rho \left[B(\dot\gamma_A(t_1)) \wedge \ldots \wedge B(\dot\gamma_A(t_k)) \, \text{Hol}(\gamma; A) \right].$$

If $n = 3$, then $A + \kappa B$ is also a connection and (2.1) is equal to the k-th Taylor coefficient in the κ-expansion of $\text{Tr}_\rho \text{Hol}(\gamma, A + \kappa B)$. When $n > 3$, then (2.1) is a differential form over LM of degree $k(n-3)$.

Unfortunately, however, (2.1) is not a good observable for the BF theory since it is not invariant under the full set of its symmetries, see below (2.3).

If we modify (2.1) so as to get a "good observable," then the relevant vacuum expectation values will produce elements of the $k(n-3)$-rd cohomology of knots (imbedded loops) as explained in the following. Observe that in the case $n = 3$ we recover the usual Vassiliev knot invariants.

2.1. **Action functional and symmetries of BF theory.** We write the action functional of the BF theory under the following

Assumption 1. We assume that the Lie algebra $\mathfrak{g}$ of G possesses a nondegenerate Ad-invariant bilinear form $\langle \, , \, \rangle$. (For example, if $\mathfrak{g}$ is semisimple, we may take the Killing form.)

We extend this form to $\Omega^*(M, \text{ad } P)$ in the usual way. Then we define

$$(2.2) \qquad\qquad S := \int_M \langle \, B \, , \, F_A \, \rangle,$$

where F_A is the curvature 2-form of A.

The Euler–Lagrange equations of motion read

$$F_A = 0, \qquad \text{d}_A B = 0;$$

thus, classically, BF theory in n dimensions describes flat connections together with covariantly closed $(n-2)$-forms. Observe that "on shell" (i.e., on the subspace of solutions) the covariant derivative is a coboundary operator.

Assumption 2. We will assume in the following that M is a compact manifold, that P is a trivial bundle and that there is a flat connection A_0 on P such that all cohomology groups $H^*_{\mathrm{d}_{A_0}}(M, \mathrm{ad}\, P)$ are trivial.

Moreover, by "on shell" we will always mean "on the subspace of those (A_0, B_0) which satisfy the equations of motion with A_0 of this kind."

The symmetries under which the action is invariant correspond to an action of the group $\mathcal{G} \rtimes_{\mathrm{Ad}} \Omega^{n-3}(M, \mathrm{ad}\, P)$, where $\mathcal{G}$ is the group of automorphisms of P (i.e., ordinary gauge transformations). Infinitesimally we have

$$\delta A = \mathrm{d}_A \xi,$$
(2.3)
$$\delta B = [\, B \,,\, \xi\,] + \mathrm{d}_A \chi,$$

with $(\xi, \chi) \in \Omega^0(M, \mathrm{ad}\, P) \rtimes_{\mathrm{ad}} \Omega^{n-3}(M, \mathrm{ad}\, P)$.

Observe that on shell this symmetries are "reducible;" i.e., a covariantly closed χ (which under our triviality assumption is of the form $\chi = \mathrm{d}_{A_0} \sigma$) acts trivially.

More precisely, each point (A_0, B_0) in the subspace of solutions has an isotropy group consisting of all covariantly closed $(n-3)$-forms of the ad type.

With our triviality assumption, the isotropy group at each point is then isomorphic to the group $\Omega^{n-4}(M, \mathrm{ad}\, P)/\mathrm{d}_{A_0}\Omega^{n-5}(M, \mathrm{ad}\, P)$.

Of course, if $n > 5$, also in this quotient there are nontrivial isotropy groups which are all isomorphic to $\Omega^{n-5}(M, \mathrm{ad}\, P)/\mathrm{d}_{A_0}\Omega^{n-6}(M, \mathrm{ad}\, P)$, and so on until we arrive at $\Omega^0(M, \mathrm{ad}\, P)$ which acts freely on $\Omega^1(M, \mathrm{ad}\, P)$.

In order to consistently gauge-fix all the symmetries, one has then to resort to the extended BRST formalism (i.e., introduce a hierarchy of ghosts for ghosts).

An additional problem is due to the fact that the isotropy groups are different off shell. So, in order to work in the Lagrangian formalism which is better suited for the perturbative expansions, one has to rely on the whole Batalin–Vilkovisky (BV) machinery as explained in Section 3.

It is known that the partition function of BF theory is related to the analytic torsion of M (see [21] for the abelian case and [3] for the non-abelian one).

In order to get other topological invariants, one has to find interesting BV closed observables. In the rest of this paper we will discuss some of them, leaving most of the technical details to [14].

Before starting with the general discussion we recall the 3-dimensional case as studied in [10, 7].

2.2. **The 3-dimensional BF theory.** Since B is a 1-form now, one can add to the pure BF action the so-called cosmological term,

$$S_3 := \frac{1}{6} \int_M \langle\, B \,,\, [\, B \,,\, B\,]\,\rangle,$$

and define

$$S_\kappa := S + \kappa^2\, S_3, \qquad \kappa \in \mathbb{R}.$$

This action is actually equal to the difference of two Chern–Simons actions evaluated at the connections $A + \kappa\, B$ and $A - \kappa\, B$. From this observation one immediately gets the infinitesimal symmetries for this theory as

$$\delta_\kappa A = \mathrm{d}_A \xi + \kappa^2\, [\, B \,,\, \chi\,],$$
(2.4)
$$\delta_\kappa B = [\, B \,,\, \xi\,] + \mathrm{d}_A \chi.$$

Another consequence is that, for any loop γ and any representation ρ of the Lie algebra $\mathfrak{g}$ of G, the Wilson loops $\mathrm{Tr}_\rho \mathrm{Hol}(\gamma; A \pm \kappa\, B)$ are observables. We then have

$$(2.5)\quad \mathcal{H}_\rho(\kappa; \gamma; A, B) = \mathrm{Tr}_\rho \mathrm{Hol}(\gamma; A+\kappa\, B) = \mathrm{Tr}_\rho \mathrm{Hol}(\gamma; A) + \sum_{k=1}^\infty \kappa^k\, h_{k,\rho}(\gamma; A, B),$$

where $h_{k,\rho}(\gamma; A, B)$ is the k-th Taylor coefficient in the κ-expansion of $\mathrm{Tr}_\rho \mathrm{Hol}(\gamma; A + \kappa\, B)$ introduced in (2.1).

By the previous discussion it follows clearly that both the even and the odd part of $\mathcal{H}$ are observables.

In order to give an explicit description of $h_{k,\rho}$, it is better to view γ as a periodic mapping from $[0, 1]$ to M. We denote by $H(\gamma; A)|_s^t$ the group element determined, in a given trivialization, by the A-parallel transport along γ from the point $\gamma(t)$ to the point $\gamma(s)$. Then we can rewrite (2.1) in the form

$$h_{k,\rho}(\gamma; A, B) = \int_{\triangle_k} \mathrm{Tr}_\rho \Big[H(\gamma; A)|_0^{t_1}\, B_1\, H(\gamma; A)|_{t_1}^{t_2}\, B_2 \cdots$$

$$\cdots B_{k-1}\, H(\gamma; A)|_{t_{k-1}}^{t_k}\, B_k\, H(\gamma; A)|_{t_k}^{1} \Big],$$

where $\triangle_k$ is the k-simplex $\{0 < t_1 < t_2 < \cdots < t_k < 1\}$ and B_i is a shorthand notation for the pullback of B via the map $\gamma_i : \triangle_k \to M$, $\gamma_i(t_1, \ldots, t_k) = \gamma(t_i)$.

Observe that each $h_{k,\rho}(\bullet; A, B)$ defines a function on the loop space $\mathrm{L}M$ of M. It is not difficult to check that—modulo the equations of motion

$$F_A + \frac{\kappa^2}{2}\, [\, B\, ,\, B\,] = 0, \qquad \mathrm{d}_A B = 0,$$

of S_κ—the functions $\mathcal{H}(\bullet; A, B)$ are locally constant on $\mathrm{L}M$.

Quantization then requires a regularization, viz., point splitting or, in a more precise formulation, the blowing up of the diagonals of configuration spaces. So one has to consider imbeddings, instead of generic loops, and to introduce a framing. The expectation values of the $\mathcal{H}$s then define locally closed functions on the space $\mathrm{Imb}_f(S^1, M)$ of framed imbeddings, i.e., framed knot invariants.

2.3. **A first glimpse to higher dimensions.** A straightforward generalization of (2.5) to higher dimensions exists as discussed at the beginning of this section. It yields forms on $\mathrm{L}M$ instead of functions since only one form degree for each field B is saturated by the integration. So now

$$h_{k,\rho}(\bullet; A, B) \in \Omega^{(n-3)k}(\mathrm{L}M).$$

Assume now that A is a flat connection and that B is covariantly closed, as classical solutions of the pure BF theory are. Then it is not difficult to check that, for any odd $n > 3$, $h_{k,\rho}$ is closed. This follows from the generalized Stokes theorem.

More precisely, $h_{k,\rho}$ is a form on $\hat{M} = \mathrm{L}M \times \mathcal{A} \times \Omega^{n-2}(M, \mathrm{ad}\, P)$, where $\mathcal{A}$ is the space of connections on M, and we restrict it to the subspace where $F_A = \mathrm{d}_A B = 0$.

Let $\hat{\mathrm{d}}$ be the differential on $\hat{M}$. In the computation of $\hat{\mathrm{d}} h_{k,\rho}$ we can switch the integral with the differential. We then get $\hat{\mathrm{d}}$ acting on a function $\eta_{k,\rho}$ on $\hat{M} \times \triangle_k$:

$$\hat{\mathrm{d}} h_{k,\rho} = \int_{\triangle_k} \hat{\mathrm{d}} \eta_{k,\rho}.$$

Let d_k be the differential on $\triangle_k$ and $\mathrm{d} := \hat{\mathrm{d}} \pm \mathrm{d}_k$ the differential on $\hat{M} \times \triangle_k$. By adding and subtracting d_k we get then

$$\hat{\mathrm{d}} h_{k,\rho} = \int_{\triangle_k} \mathrm{d}\eta_{k,\rho} \mp \int_{\triangle_k} \mathrm{d}_k \eta_{k,\rho}.$$

The first term on the r.h.s. is then easily seen to vanish in our hypotheses $F_A = \mathrm{d}_A B = 0$ since

$$\mathrm{d}H(\gamma; A)|_s^t = -A(s)\, H(\gamma; A)|_s^t + H(\gamma; A)|_s^t\, A(t)$$

when A is flat.

The second term can then be computed using the Stokes theorem. The codimension-one boundaries of the simplex corresponding to the collapse of consecutive points yield terms containing B^2 which vanishes for dimensional reasons ($n > 4$). The remaining codimension-one boundaries correspond to $t_0 = 0$ or $t_k = 1$. It is not difficult to check, using the cyclic property of the trace, that these terms cancel each other.

If the dimension n is even, $n > 4$, the only problem in the above discussion arises at the last step since, for k even, the two terms coming from $t_0 = 0$ and $t_k = 1$ sum up instead of canceling each other. For k odd however everything works as before.

Similar computations allow to show that the $h_{k,\rho}$ are invariant (modulo exact forms on $\mathrm{L}M$) under the symmetries (2.3) either if n is odd and greater then 5 or if n is even and greater than 4 and k is odd.

In sections 4 and 5 we will describe how this discussion can be extended "off shell" and how the cases $n = 4$ and $n = 5$ will be included.

3. The BV quantization of BF theories

3.1. **The BRST operator.** In order to deal with the symmetries (2.3) of (2.2) in the functional integral, one has to introduce the BRST operator

$$(3.1) \qquad \begin{aligned} \delta_{\mathrm{BRST}} A &= \mathrm{d}_A c, \\ \delta_{\mathrm{BRST}} B &= [\, B\, ,\, c\,] + \mathrm{d}_A \tau_1, \end{aligned}$$

where c and τ_1 are ghosts, i.e., forms on the space of fields with values in $\Omega^0(M, \mathrm{ad}\, P)$ and $\Omega^{n-3}(M, \mathrm{ad}\, P)$ respectively. As usual in gauge theories one also defines

$$\delta_{\mathrm{BRST}} c = -\frac{1}{2}[c\, ,\, c].$$

Because of the on-shell reducibility, one has then to introduce ghosts for ghosts τ_k with values in $\Omega^{n-2-k}(M, \mathrm{ad}\, P)$, $k = 1, \ldots, n-2$, with ghost number equal to $k \mod 2$ and extended BRST operator

$$\begin{aligned} \delta_{\mathrm{BRST}} \tau_k &= (-1)^k [\, \tau_k\, ,\, c\,] + \mathrm{d}_A \tau_{k+1}, \qquad k = 1, \ldots, n-3, \\ \delta_{\mathrm{BRST}} \tau_{n-2} &= (-1)^n [\, \tau_{n-2}\, ,\, c\,]. \end{aligned}$$

3.2. **The BV formalism.** It is not difficult to check that $\delta_{\mathrm{BRST}}^2 = 0 \mod F_A$.

The Batalin–Vilkovisky method allows then for the construction of a nilpotent operator δ_{BV} that extends δ_{BRST} off shell.

To do so, one first introduces a partner ϕ_α^+ with values in $\Omega^*(M, \mathrm{ad}\, P)$ for any field or ghost $\phi^\alpha = A, B, c, \tau_1, \ldots, \tau_{n-2}$ with the following rules:

- The ghost number of ϕ_α^+ is minus the ghost number of ϕ^α, minus one.
- The form degree of ϕ_α^+ is n minus the form degree of ϕ^α.

Remark 3.1. In general, the antifields are dual to the corresponding fields. Of course some isomorphisms may be used to identify certain spaces. For example here we have preferred to identify the Lie algebra $\mathfrak{g}$ with its dual (using the bilinear from $\langle\,,\,\rangle$) so that also the antifields take values in the space of tensorial forms of the ad type. This will be particularly useful, e.g., in equations (3.5) and (3.6).

In the original formulation of Batalin and Vilkovisky [2] one also identifies forms of complementary degree using a Hodge operator. Since we do not want to introduce a metric here, we prefer to avoid this identification. As a consequence, our BV antibracket (3.2) will be of the form described in [13] instead of the original one.

Remark 3.2 (Sign convention). We follow here the usual convention for the sign rules related to the double grading given by the form degree $\deg$ and the ghost number gh.

Namely, in the case of homogenous forms α and β of the ad-type we have

$$[\,\alpha\,,\,\beta\,] = -(-1)^{\deg\alpha\,\deg\beta+\mathrm{gh}\,\alpha\,\mathrm{gh}\,\beta}[\,\beta\,,\,\alpha\,].$$

Moreover, in the case of homogenous forms α and β taking values in a commutative algebra (e.g., $\mathbb{R}$) we have

$$\alpha\wedge\beta = (-1)^{\deg\alpha\,\deg\beta+\mathrm{gh}\,\alpha\,\mathrm{gh}\,\beta}\beta\wedge\alpha.$$

Next we define the BV bracket of two functionals F and G. We use throughout Einstein's convention over repeated indices and set

(3.2)

$$(\,F\,,\,G\,) := \int_M \left\langle F\,\frac{\overleftarrow{\partial}}{\partial\phi^\alpha}\,,\,\frac{\overrightarrow{\partial}}{\partial\phi^+_\alpha}\,G\right\rangle - (-1)^{\deg\phi^\alpha\,(n+1)}\left\langle F\,\frac{\overleftarrow{\partial}}{\partial\phi^+_\alpha}\,,\,\frac{\overrightarrow{\partial}}{\partial\phi^\alpha}\,G\right\rangle,$$

where the left and right functional derivatives are given by the following formula:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}F(\phi^\alpha + t\,\rho^\alpha) = \int_M\left\langle\rho^\alpha\,,\,\frac{\overrightarrow{\partial}}{\partial\phi^\alpha}\,F\right\rangle = \int_M\left\langle F\,\frac{\overleftarrow{\partial}}{\partial\phi^\alpha}\,,\,\rho^\alpha\right\rangle;$$

we proceed similarly for the antifields.

As usual, the space of functionals with BV bracket is a Gerstenhaber algebra [18].

Finally the BV operator is defined by

(3.3) $$\delta_{\mathrm{BV}} := (\,S_{\mathrm{BV}}\,,\,\ )$$

where S_{BV} is a solution of the master equation

$$(\,S_{\mathrm{BV}}\,,\,S_{\mathrm{BV}}\,) = 0$$

such that $S_{\mathrm{BV}}|_{\phi^+_\bullet=0} = S$.

3.3. **The BV action for BF theories.** The BV action S_{BV} corresponding to the BF action (2.2) can be written as

(3.4) $$S_{\mathrm{BV}} = \int_M \langle\!\langle\, \mathsf{B}\,;\,\mathsf{F}_\mathsf{A}\,\rangle\!\rangle$$

where the notations are as follows:

- The *dot product* is just the wedge product between forms taking values in an associative algebra—e.g., $\mathbb{R}$ or $\mathfrak{g}$ itself if it associative as in Section 5—but with a shifted degree; viz., for two homogenous forms α and β we set

$$\alpha\cdot\beta := (-1)^{\mathrm{gh}\,\alpha\,\deg\beta}\alpha\wedge\beta,$$

where gh denotes the ghost number.

We extend then the bilinear form $\langle\,,\,\rangle$ to forms with shifted degree by setting

$$\langle\!\langle\,\alpha\,;\,\beta\,\rangle\!\rangle := (-1)^{\mathrm{gh}\,\alpha\,\deg\beta}\,\langle\,\alpha\,,\,\beta\,\rangle\,.$$

Similarly we define the *dot Lie bracket* for two homogeneous forms of the ad type by

$$[\![\,\alpha\,;\,\beta\,]\!] := (-1)^{\mathrm{gh}\,\alpha\,\deg\beta}\,[\,\alpha\,,\,\beta\,].$$

These definitions are then extended by linearity.

An easy check shows that the dot product in the case of a commutative algebra and the dot Lie bracket are, respectively, a graded commutative product and a graded Lie bracket with respect to a new grading called the *total degree* that is defined as the form degree plus the ghost number. Moreover, d_{A_0} is still a differential for the dot algebras.

- The "super B-field" B is defined by

$$(3.5)\qquad \mathsf{B} = \sum_{k=1}^{n-2}(-1)^{\frac{k(k-1)}{2}}\tau_k + B + (-1)^n A^+ + c^+ \in \Omega^*(M,\mathrm{ad}\,P),$$

and has total degree equal to $n-2$.

- The "supercurvature" F_A of the "superconnection"

$$(3.6)\qquad \mathsf{A} = (-1)^{n+1}c + A + (-1)^n B^+ + \sum_{k=1}^{n-2}(-1)^{n(k+1)+\frac{k(k-1)}{2}}\tau_k^+$$

is given by the usual formula. In order to write it down, it is better to choose a background connection A_0 and to define the tensorial form $\mathsf{a} = \mathsf{A} - A_0$ of total degree one. Then

$$\mathsf{F}_\mathsf{A} = F_{A_0} + \mathrm{d}_{A_0}\mathsf{a} + \frac{1}{2}[\![\,\mathsf{a}\,;\,\mathsf{a}\,]\!].$$

In general we will choose A_0 to be a flat connection as in Assumption 2.

- By $\int_M$ we then mean the integral of all the terms of form degree equal to n. Observe that as a consequence S_{BV} has then ghost number zero.

Remark 3.3. We may observe that there is a superspace formulation of (3.4) obtained by introducing superpartners to the coordinates of M and redefining A and B accordingly. In this way we would follow the pattern described in [16].

Special cases were already discussed in [6, 13] (two dimensions) and [11, 8] (four dimensions). See also [26, 1] for the case of the 3-dimensional Chern–Simons theory.

For later purposes—viz., in order to define loop observables as in the following sections—it is however better to work in our setting.

We have the following general result [14]:

Theorem 1. *The action S_{BV} satisfies the master equation in any dimension.*

We conclude this section by giving the explicit action of the BV operator (3.3) on the "superfields" A and B. In order to give neater formulae, it is better to define a new BV operator with shifted degree; viz., for a homogeneous form α, we set

$$\boldsymbol{\delta}\alpha := (-1)^{\deg\alpha}\delta_{\mathrm{BV}}\alpha.$$

One can show that $\boldsymbol{\delta}$ is a differential for the dot algebras and that it anticommutes with d_{A_0}.

Then we obtain [14]

$$\boldsymbol{\delta}\mathsf{A} = (-1)^n \, \mathsf{F_A},$$
$$\boldsymbol{\delta}\mathsf{B} = (-1)^n \, \mathsf{d_A B},$$

(3.7)

with

$$\mathsf{d_A B} = \mathsf{d}_{A_0}\mathsf{B} + [\![\, \mathsf{a} \, ; \, \mathsf{B} \,]\!].$$

Upon using the above equations, we can then prove Thm. 1 by simply checking that $\boldsymbol{\delta}S_{\mathrm{BV}} = 0$, as follows from the the ad-invariance of $\langle\, , \,\rangle$ and from the Stokes theorem.

3.4. **The BV Laplace operator and the BV observables.** In the quantum version of the BV formalism—i.e., when dealing with functional integrals with weight $\exp(\mathrm{i}/\hbar)S$—one has then to introduce the so-called BV Laplace operator Δ_{BV} and to verify that the *quantum* master equation

$$(\, S_{\mathrm{BV}} \, , \, S_{\mathrm{BV}} \,) - 2\mathrm{i}\hbar\Delta_{\mathrm{BV}}S_{\mathrm{BV}} = 0$$

is satisfied.

The very definition of Δ_{BV} relies on a regularization of the theory, which we do not discuss here. We only recall the formal properties of Δ_{BV}; viz.:

1. Δ_{BV} is a coboundary operator on the space of functionals;
2. for any two functionals F and G,

(3.8) $$\Delta_{\mathrm{BV}}(F\,G) = (\Delta_{\mathrm{BV}}F)\,G + (-1)^{\mathrm{gh}\,F}\,F\Delta_{\mathrm{BV}}G + (-1)^{\mathrm{gh}\,F}\,(\,F\,,\,G\,).$$

The space of functionals with the BV bracket and the BV Laplacian is a so-called BV algebra.

To give an explicit definition of the BV Laplacian one has to introduce some extra structures (e.g., a Riemannian metric on M) and a regularization. The main property however is that the BV Laplace operator contracts each field with the Hodge dual of the corresponding antifield at the same point in M.

Under our assumptions, one can then prove [14] that $\Delta_{\mathrm{BV}}S_{\mathrm{BV}} = 0$ for S_{BV} in (3.4).

So S_{BV} is also a solution of the quantum master equation. This implies that its partition function is independent of the choice of gauge fixing.

A consequence of the properties of the BV operators is that the operator

$$\Omega_{\mathrm{BV}} := \delta_{\mathrm{BV}} - \mathrm{i}\hbar\Delta_{\mathrm{BV}}$$

is a coboundary operator iff S_{BV} satisfies the quantum master equation.

The main statement in the BV formalism is that the Ω_{BV}-cohomology of ghost number zero yields all the meaningful observables. More precisely, this means that the expectation value of an Ω_{BV}-closed functional is independent of the gauge fixing and that the expectation value of an Ω_{BV}-exact functional (or of a functional of ghost number different from zero) vanishes.

In the next sections we will discuss some BV observables of BF theories associated to $\mathrm{L}M$ (or better to $\mathrm{Imb}_f(S^1, M)$). To do so, it is however better to use shifted versions of the operators Ω_{BV} and Δ_{BV} as well, viz.:

$$\boldsymbol{\Omega}\alpha := (-1)^{\deg\alpha}\Omega_{\mathrm{BV}}\alpha,$$
$$\boldsymbol{\Delta}\alpha := (-1)^{\deg\alpha}\Delta_{\mathrm{BV}}\alpha.$$

4. GENERALIZED WILSON LOOPS IN ODD DIMENSIONS

At this point we are ready to define the correct generalization of the observables $\mathcal{H}$ defined in (2.5).

Formally the new observable is still the trace of the "holonomy of $\mathsf{A} + \kappa\,\mathsf{B}$"

$$(4.1) \qquad \mathcal{H}_\rho(\kappa; \mathsf{A}, \mathsf{B}) := \mathrm{Tr}_\rho \, \mathsf{Hol}(\mathsf{A} + \kappa\,\mathsf{B}) \in \Omega^*(LM),$$

where the "holonomy" Hol is now defined in terms of iterated integrals as follows: First we write $\mathsf{A} = A_0 + \mathsf{a}$. Then we set

$$\mathrm{Tr}_\rho \, \mathsf{Hol}(\mathsf{A} + \kappa\,B) := \mathrm{Tr}_\rho \, \mathrm{Hol}(A_0) + \sum_{l=1}^\infty h_{l,\rho}(A_0, \mathsf{a} + \kappa\,\mathsf{B}),$$

where

$$h_{l,\rho}(A_0, \mathsf{a} + \kappa\,\mathsf{B}) = \int_{\triangle_l} \mathrm{Tr}_\rho \left[H(A_0)|_0^{t_1} \cdot (\mathsf{a}_1 + \kappa\,\mathsf{B}_1) \cdot H(A_0)|_{t_1}^{t_2} \cdot \right.$$
$$\left. \cdot (\mathsf{a}_2 + \kappa\,\mathsf{B}_2) \cdots H(A_0)|_{t_{l-1}}^{t_l} \cdot (\mathsf{a}_l + \kappa\,\mathsf{B}_l) \cdot H(A_0)|_{t_l}^1 \right].$$

Here a_i and B_i are shorthand notations for the pullbacks of a and B via $\mathrm{ev}_i : LM \times \triangle_l \to M$, $(\gamma; t_1, \ldots, t_l) \mapsto \gamma(t_i)$. Moreover, the $H(A_0)|_s^t$'s denote the group elements associated to parallel transports as functions on LM.

Remark 4.1. The above integrals should be better viewed as integrations along the fiber of the trivial bundles $\triangle_l \times LM \to LM$. That is, the integrals are zero whenever the form degree is less than the dimension of the simplex and yield a form on LM whenever the form degree exceeds the dimension of the simplex.

Also observe that $\mathcal{H}$ is a sum of terms with different ghost number and different form degree on LM.

Of course, we cannot expect $\mathcal{H}$ to be an observable for the pure BF theory. We can however consider a fake "higher dimensional BF theory with cosmological term" as follows: We first define

$$(4.2) \qquad \mathsf{S}_3(\mathsf{B}) := \frac{1}{6} \int_M \langle\!\langle \, \mathsf{B} \, ; \, [\![\, \mathsf{B} \, ; \, \mathsf{B} \,]\!] \, \rangle\!\rangle,$$

where again we consider only the terms of form degree equal to n, so S_3 has ghost number $2(n-3)$. Then we consider the functional

$$(4.3) \qquad \widehat{\mathcal{H}}_\rho(\hbar, \kappa; \mathsf{A}, \mathsf{B}) := \left\{ \exp[(\mathrm{i}/\hbar)\,\kappa^2\,\mathsf{S}_3(\mathsf{B})] \cdot \mathcal{H}_\rho(\kappa; \mathsf{A}, \mathsf{B}) \right\}_0,$$

where $\{\}_0$ means taking the terms with ghost number zero.

The functional $\widehat{\mathcal{H}}_\rho$ is well-defined for any loop in M. However, in order to avoid problems with the BV Laplace operator Δ_{BV}, we must restrict ourselves to the space of framed imbeddings $\mathrm{Imb}_f(S^1, M)$.

Theorem 2. *For any κ and ρ and any odd dimension n, $\widehat{\mathcal{H}}$, as a functional taking values in the forms on $\mathrm{Imb}_f(S^1, M)$, is Ω_{BV}-closed modulo* d*-exact forms and* d*-closed modulo Ω_{BV}-exact terms. In other words, $[\widehat{\mathcal{H}}]$ is an $H^*(\mathrm{Imb}_f(S^1, M))$-valued observable.*

Proof (Sketch). The main idea of the proof relies on the identity

$$(4.4) \quad \mathbf{\Omega} \left\{ \exp[(\mathrm{i}/\hbar)\,\kappa^2\,\mathsf{S}_3(\mathsf{B})] \cdot \mathcal{H}_\rho(\kappa; \mathsf{A}, \mathsf{B}) \right\} = \exp[(\mathrm{i}/\hbar)\,\kappa^2\,\mathsf{S}_3(\mathsf{B})] \cdot \boldsymbol{\delta}_\kappa \mathcal{H}_\rho(\kappa; \mathsf{A}, \mathsf{B}),$$

where $\boldsymbol{\delta}_\kappa$ is the following coboundary operator:

$$\boldsymbol{\delta}_\kappa \mathsf{A} = -\mathsf{F}_\mathsf{A} - \frac{\kappa^2}{2}\,[\![\,\mathsf{B}\,;\,\mathsf{B}\,]\!],$$
$$\boldsymbol{\delta}_\kappa \mathsf{B} = -\mathrm{d}_\mathsf{A}\mathsf{B}.$$

Observe that for $n \neq 3$, $\boldsymbol{\delta}_\kappa$ is a differential only for the $\mathbb{Z}_2$-reduction of the graded algebra of functionals.

Using $\boldsymbol{\delta}_\kappa$ is like working with a cosmological term, and, upon using the generalized Stokes theorem, one gets

$$(\mathrm{d} + \boldsymbol{\delta}_\kappa)\,\boldsymbol{\mathcal{H}}_\rho(\kappa; \mathsf{A}, \mathsf{B}) = 0,$$

which proves the theorem.

Equation (4.4) is a consequence of (3.8) and of the following identities:

$$\boldsymbol{\delta}\mathsf{S}_3 = 0, \quad \boldsymbol{\Delta}\exp[(\mathrm{i}/\hbar)\,\kappa^2\,\mathsf{S}_3(\mathsf{B})] = 0, \quad \boldsymbol{\Delta}\boldsymbol{\mathcal{H}}_\rho = 0.$$

The first identity follows from (3.7), from the fact that $\langle\,,\,\rangle$ is ad-invariant and from Stokes theorem.

The second identity holds since S_3 depends only on B and as a consequence of the already discussed property according to which the BV Laplace operator contracts each field with the Hodge dual (for some Riemannian metric on M) of the corresponding antifield at the same point in M.

The last identity is "formally" (i.e., modulo regularization problems for Δ_{BV}) true if γ does not have transversal self-intersections, for the same reason as above. However, in order to rely upon this last identity confidently, we must then restrict ourselves to framed imbeddings and put each component of A on the imbedding and each component of B on its companion (as done in [10]).   □

As a consequence, the expectation value of $\widehat{\boldsymbol{\mathcal{H}}}$ is (up to anomalies) a cohomology class on the space of (framed) imbeddings of S^1 into M.

Remark 4.2. If we set all the antifields to zero, $\widehat{\boldsymbol{\mathcal{H}}}$ reduces to a sum of $h_{k,\rho}(A, B)$'s; so it is the off-shell generalization we were looking for in subsection 2.3.

Moreover, the expectation value of $\widehat{\boldsymbol{\mathcal{H}}}$ w.r.t. the pure BF theory is in three dimensions the same as the expectation value of $\mathcal{H}$ in the BF theory with cosmological term.

Remark 4.3. $\widehat{\boldsymbol{\mathcal{H}}}$ is a genuine quantum observable since its limit for $\hbar \to 0$ is not defined.

However, one might replace κ with $\hbar\kappa$. In this way, $\widehat{\boldsymbol{\mathcal{H}}}$ becomes a formal power series in $\hbar$. The zeroth order term is just $\mathrm{Tr}_\rho \mathrm{Hol}(A)$.

Since $\widehat{\boldsymbol{\mathcal{H}}}$ is an observable for any κ, so is its odd part (in κ) $\widehat{\boldsymbol{\mathcal{H}}}^o$. It is not difficult to see that $\widehat{\boldsymbol{\mathcal{H}}}_\rho^o(\hbar, \hbar\kappa; \mathsf{A}, \mathsf{B})/\hbar$ is as well a formal power series in $\hbar$ and that its zeroth-order term is the observable

$$\mathsf{h}_{1,\rho}(\mathsf{A}, \mathsf{B}) := \frac{\mathrm{d}}{\mathrm{d}\kappa}\Big|_{\kappa=0} h_{1,\rho}(A_0, \mathsf{a} + \kappa B),$$

which is the off-shell extension of $h_{1,\rho}(A, B)$.

Therefore, the observables $\widehat{\boldsymbol{\mathcal{H}}}_\rho(\hbar, \hbar\kappa; \mathsf{A}, \mathsf{B})$ and $\widehat{\boldsymbol{\mathcal{H}}}_\rho^o(\hbar, \hbar\kappa; \mathsf{A}, \mathsf{B})/\hbar$ are nontrivial quantum deformations of, respectively, the ordinary Wilson loop and $\mathsf{h}_{1,\rho}$.

5. OTHER LOOP OBSERVABLES

In order to generalize some of the results of the previous section and in order to define more general observables we make the following

Assumption 3. We assume that the Lie algebra $\mathfrak{g}$ is obtained from an associative algebra with trace Tr (e.g., we may take $\mathfrak{g} = \mathfrak{gl}(N)$ with the usual trace of matrices). In this case, we assume that our ad-invariant bilinear form $\langle\, \xi\, ,\, \eta\, \rangle$ is given by $\mathrm{Tr}(\xi\,\eta)$ and, according to Assumption 1, we further assume that it is nondegenerate. Moreover, we consider only representations ρ of $\mathfrak{g}$ as an associative algebra.

5.1. Generalized Wilson loops in even dimensions.
The observable defined in the previous section does not work in even dimensions essentially because the "cosmological term" S_3 vanishes when B has even total degree.

We can cure this problem thanks to Assumption 3 by defining instead

$$\mathsf{O}_3(\mathsf{B}) := \frac{1}{3}\int_M \mathrm{Tr}(\mathsf{B}\cdot\mathsf{B}\cdot\mathsf{B}).$$

We have already seen in subsection 2.3 that the even part of $\mathcal{H}$ does not work. So we consider only

$$(5.1)\qquad \widehat{\mathcal{H}}_\rho^{\,o}(\hbar,\kappa;\mathsf{A},\mathsf{B}) := \big\{\exp[(\mathrm{i}/\hbar)\,\kappa^2\,\mathsf{O}_3(\mathsf{B})]\cdot\mathcal{H}_\rho^o(\kappa;\mathsf{A},\mathsf{B})\big\}_0,$$

where $\mathcal{H}^o$ is the odd part of $\mathcal{H}$, which is defined exactly as in the odd-dimensional case—see equations (4.1) and following.

We have then the following analogue (with analogous proof) of Thm. 2:

Theorem 3. *For any κ and ρ and any even dimension n, $[\widehat{\mathcal{H}}^{\,o}]$ is an $H^*(\mathrm{Imb}_f(S^1,M))$-valued observable.*

Remark 5.1. Similarly to what happens in the odd-dimensional case, $\widehat{\mathcal{H}}^{\,o}$ reduces to a sum of $h_{2k+1,\rho}$ as the antifields are set to zero.

Moreover, $\widehat{\mathcal{H}}_\rho^{\,o}(\hbar,\hbar\kappa;\mathsf{A},\mathsf{B})/\hbar$ is still a nontrivial quantum deformation of $h_{1,\rho}$.

However, we do not find in even dimensions a nontrivial quantum deformation of the ordinary Wilson loop $\mathrm{Tr}_\rho\,\mathrm{Hol}$.

5.2. Loop observables with more then cubic interactions.
The "cosmological terms" S_3 and O_3 give rise, in the perturbative expansion, to trivalent vertices.

If we work with Assumption 3, we can define more general interaction terms:

$$\mathsf{O}_r(\mathsf{B}) := \frac{1}{r}\int_M \mathrm{Tr}\,\mathsf{B}^r.$$

Observe that in odd dimensions O_r vanishes if r is even.

Next, for any two given sequences $\boldsymbol{\mu} = \{\mu_1, \mu_2, \dots\}$ and $\boldsymbol{\lambda} = \{\lambda_1, \lambda_2, \dots\}$, we define

$$(5.2)$$
$$\widetilde{\mathcal{H}}_\rho(\hbar,\boldsymbol{\mu},\boldsymbol{\lambda};\mathsf{A},\mathsf{B}) := \left\{\exp\left[(\mathrm{i}/\hbar)\sum_{r=1}^\infty \mu_r\,\mathsf{O}_{r+1}(\mathsf{B})\right]\cdot\mathrm{Tr}_\rho\,\mathrm{Hol}\left(\mathsf{A}+\sum_{s=1}^\infty \lambda_s\,\mathsf{B}^s\right)\right\}_0.$$

We then denote by $\widetilde{\mathcal{H}}_\rho^{\,o}$ the odd part of $\widetilde{\mathcal{H}}_\rho$ under $\boldsymbol{\lambda}\to-\boldsymbol{\lambda}$.

We have then

Theorem 4. *In odd dimensions, $[\widetilde{\boldsymbol{\mathcal{H}}}_\rho]$ is an $H^*(\mathrm{Imb}_f(S^1, M))$-valued observable whenever the following conditions are satisfied*

$$\mu_{2l-1} = \lambda_{2l} = 0, \quad \forall l,$$

$$\mu_{2l} = \sum_{\substack{i,j\geq 0 \\ i+j=l-1}} \lambda_{2i+1}\,\lambda_{2j+1}, \quad \forall l.$$

In even dimensions, $[\widetilde{\boldsymbol{\mathcal{H}}}_\rho^{\,o}]$ is an $H^*(\mathrm{Imb}_f(S^1, M))$-valued observable whenever the following conditions are satisfied

$$\mu_l = \sum_{\substack{i,j\geq 1 \\ i+j=l}} \lambda_i\,\lambda_j, \quad \forall l.$$

The proof is a direct generalization of the proof of Thm. 2.

6. Conclusions

In this paper we have defined some $H^*(\mathrm{Imb}_f(S^1, M))$-valued observables for BF theories on a trivial principal G-bundle $P \to M$ associated to any representation of the Lie algebra $\mathfrak{g}$.

Our ideas extend naturally to nontrivial bundles as well, though we did not consider this extension here for the sake of simplicity.

The expectation values of these observables define then classes in the cohomology of the space of framed imbeddings of S^1 into M, which we have assumed to be compact in order to simplify the discussion.

Of course a very interesting case is $M = \mathbb{R}^n$, which is not compact. The only extra technical point here is that one has to specify the correct behavior at infinity of all the fields and antifields. This done, the trivial connection satisfies the hypotheses of Assumption 2.

The perturbative expansion of the expectation values in the case $M = \mathbb{R}^n$ around the trivial connection is then obtained in terms of the configuration space integrals discussed in [9], where however only the framing-independent coholomogy classes were considered explicitly.

Notice that in this paper we have *not* defined observables with trivalent interactions and an even number of B-fields placed on the imbedding in the even-dimensional case. Thus, we cannot obtain the nontrivial class of imbeddings represented by the diagram cocycle of Figure 4 in [9]. This suggests that there might exist other observables than those we have considered here.

On the other hand, one may use the combinatorics of this quantum field theory to obtain new nontrivial diagram cocycles.

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